Similarly, for a supercharged engine

$$t_{FC} = \frac{2\beta}{a} \ln \frac{ae^{-h_1/2\beta} - b}{ae^{-h_2/2\beta} - b}, \quad \text{for} \quad h_2 \leq h_{cr} \tag{26}$$

$$t_{FC} = \frac{2\beta A^*}{3b}\left[\sqrt{3}\left(\tan^{-1}\frac{2e^{-h_2/2\beta} + A^*}{A^*\sqrt{3}} - \tan^{-1}\frac{2e^{-h_1/2\beta} + A^*}{A^*\sqrt{3}}\right) + \frac{1}{2}\ln\frac{e^{-h_2/\beta} + A^*e^{-h_2/2\beta} + A^{*2}}{e^{-h_1/\beta} + A^*e^{-h_1/2\beta} + A^{*2}} - \ln\frac{e^{-h_2/2\beta} - A^*}{e^{-h_1/2\beta} - A^*}\right], \quad \text{for} \quad h_1 \geq h_{cr} \tag{27}$$

Fuel Consumption

The amount of fuel consumed during climb from h_1 to h_2 is expressed in nondimensional form ζ which is called the climb-fuel weight fraction. Its value ζ_{FC} for the fastest climb is obtained from Eq. (4) as

$$\zeta_{FC} = 1 - \exp\left[-\hat{c}\int_{h_1}^{h_2} \frac{P_e/W}{(R/C)_m}\,dh\right], \qquad h_2 \geq h_1 \tag{28}$$

For aspirated engines, the use of Eqs. (5), (17), and (8) allows the integration of the right side of the above equation in analytical form. Therefore, the above relation becomes

$$\zeta_{FC} = 1 - \exp\left[\frac{2\beta\hat{c}(P_{e,SSL}/W)}{3a} \ln \frac{ae^{-3h_2/2\beta} - b}{ae^{-3h_1/2\beta} - b}\right] \tag{29}$$

Similarly, for supercharged engines

$$\zeta_{FC} = 1 - \exp\left[\frac{2\beta\hat{c}(P_{e,SSL}/W)}{a} \ln \frac{ae^{-h_2/2\beta} - b}{ae^{-h_1/2\beta} - b}\right] \quad \text{for} \quad h_2 \leq h_{cr} \tag{30}$$

$$\zeta_{FC} = 1 - \exp\left[\frac{2\beta\hat{c}(P_{e,SSL}/W)}{3a} \ln \frac{ae^{-3h_2/2\beta} - b\sigma_{cr}}{ae^{-3h_1/2\beta} - b\sigma_{cr}}\right] \quad \text{for} \quad h_1 \geq h_{cr} \tag{31}$$

The notations a and b are obtained from Eq. (11), and σ_{cr} is also supposed to be a known quantity before proceeding for calculations of the climb performance of a piston-prop aircraft.

Application to an Aircraft

A piston-prop aircraft of total weight of 7350 N (1652.3 lb), wing area of 11.2 m^2 (120.5 ft^2) and engine power of 75 kW (100.6 hp) has the parabolic drag polar $C_D = 0.032 + 0.14C_L^2$. Its power-specific fuel consumption is 0.55×10^{-6} N/s W (0.33 lb/h hp), propeller efficiency of 0.85 and a critical altitude at 6 km. Find the range, endurance, and fuel weight fraction as the aircraft passes through different known altitudes during the fastest climb in standard atmosphere, both for aspirated and supercharged engines.

The above aircraft has wing loading (W/S) of 656.25 N/m^2, engine power-to-weight ratio $(P_{e,SSL}/W)$ of 10.204 W/N, and maximum aerodynamic efficiency E_m of 7.47. Since the flight is confined in the troposhere, $\beta = 9296$ m. From the standard atmospheric table, $\rho_{SSL} = 1.226$ kg/m^3 and $\sigma_{cr} = \rho_{cr}/\rho_{SSL} = 0.538$ at the critical altitude of 6000 m. The use of Eq. (9) gives $V_{P\min,SSL} = 35.96$ m/s and from the two equations of relation (11), $a = 8.673$ and $b = 5.558$.

It is assumed that the steady straight climbing flight has started from the ground at sea level itself, therefore, $h_1 = 0$ in Eqs. (21), (22), (25), (26), (29), and (30). The h_2 would be regarded as the known variable h ($\geq h_1$). Using Eqs. (21), (25), and (29), respectively, x_{FC}, t_{FC}, and ζ_{FC} are obtained at different altitudes for aspirated piston-prop engine aircraft and these are shown in Fig. 1 as the three lower curves. In the case of supercharged (turbocharged) engines, the x_{FC}, t_{FC}, and ζ_{FC} are obtained from Eqs. (22), (26), and (30) for below the critical altitude, and from Eqs. (23), (27), and (31) for above the critical altitude. These results are also shown in Fig. 1 and they form the three upper curves of the figure. The curves for both aspirated and supercharged engines flatten out as they approach their ceiling altitudes because of the decrease in rate of climb.

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Special Rotation Vectors: A Means for Transmitting Quaternions in Three Components

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Introduction

THE use of quaternions[1–3] for computing and recording the orientation of a body in three dimensions[4,5] has been gaining acceptance. This method has clear advantages over traditional Euler angles.[6–8] The differential equations governing the time development of quaternions are free of singularities for all orientations; trigonometric functions are eliminated from the differential equations as well as from the process of constructing a rotation matrix. In terms of computational load, these advantages clearly outweigh the need to treat four, rather than three, components and to enforce an auxiliary condition against truncation errors.

There is one situation, however, where the trade between quaternions and Euler angles is not clearly decided. This is the case when orientation information is passed over a data link. Here, quaternions require an additional one-third of bandwidth. In some cases bandwidth is a scarcer commodity than computational throughput, and this might tip the balance in favor of Euler angles.

In this Note we introduce a scheme for eating your cake and having it too. We show how it is possible to capture quaternion style orientation information in a three component special rotation vector (SRV). The mathematical theory of SRVs and their relationship to the group of rotations in three dimensions will be covered elsewhere.[9] The purpose of this Note is to provide the practical rules for using SRVs for transmitting quaternion-style orientation information. The method of using quaternions in this context is outlined in Appendix A. Between the body of the text and Appendix A this Note is self-contained. It can be understood and applied without reading Ref. 9. The correspondence between SRVs and rotations is specified in Appendix B.

Quaternions consist of a vector part $\boldsymbol{Q}$ that has three components and a scalar part Q_0. The four components are used in Eq. (A14) to produce the rotation matrix. They are propagated in time by the differential Eqs. (A18) and (A19).

Now suppose that two computers, both using quaternions to represent orientation, need to pass this information to each

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other over a data link. This can be done sending only three parameters by the following steps:

1) The sending computer converts the quaternion into a three components SRV, $\boldsymbol{R}$ by

$$\boldsymbol{R} = \begin{cases} \boldsymbol{Q}, & \text{if} \quad Q_0 \geq 0 \\ -\boldsymbol{Q}, & \text{if} \quad Q_0 < 0 \end{cases} \tag{1}$$

2) The three components of $\boldsymbol{R}$ are passed over the link.

3) The receiving computer reconstructs a quaternion Q' from the SRV by

$$\boldsymbol{Q}' = \boldsymbol{R}; \qquad Q_0' = +(1 - |\boldsymbol{R}|^2)^{1/2} \tag{2}$$

The positive square root is used in evaluating Q_0'.

The relationship of the new quaternion Q' to the original Q is either $Q_0' = Q_0$, $\boldsymbol{Q}' = \boldsymbol{Q}$ (when $Q_0 \geq 0$) or $Q_0' = -Q_0$, $\boldsymbol{Q}' = -\boldsymbol{Q}$ (when $Q_0 < 0$). However, both $(Q_0, \boldsymbol{Q})$ and $(-Q_0, -\boldsymbol{Q})$ correspond to the same orientation and produce the same rotation matrix when substituted in Eq. (A21) or (A22). (In the case of quaternions, Q_0 is determined by $\boldsymbol{Q}$ up to a sign. It is enough to transmit the three components of $\boldsymbol{Q}$ with full accuracy and one extra bit specifying the sign of Q_0. It is thus possible, in principle, to transmit full quaternion information in three variables plus one bit. However, in practice the transmission of odd bits is awkward. SRVs, in reducing the transmitted information to exactly three variables, are more satisfactory.)

Appendix A: Representation of Orientation by Quaternions

Quaternions are generalized numbers using four units: the ordinary number 1 and three additional units $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ that satisfy

$$i^2 = j^2 = k^2 = -1 \tag{A1}$$

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k} \tag{A2}$$

$$\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i} \tag{A3}$$

$$\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j} \tag{A4}$$

A general quaternion has the form

$$Q = Q_0 + Q_x i + Q_y j + Q_z k \tag{A5}$$

with the numbers Q_0, Q_x, Q_y, Q_z being the "components" of the quaternion. The units $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ may be thought of as the unit vectors in the directions of the three coordinate axes. The part involving $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ may be considered a vector $\boldsymbol{Q}$. Q_0 and $\boldsymbol{Q}$ are called the number part and the vector part of the quaternion, respectively. The quaternion may be conveyed as a combination

$$Q = (Q_0, \boldsymbol{Q}) \tag{A6}$$

of a number and a vector. However, in the formalism of quaternions vectors and numbers can be mixed in addition as well as multiplication. Another way to write the general quaternion of Eqs. (A5) and (A6) is

$$Q = Q_0 + \boldsymbol{Q} \tag{A7}$$

Quaternions can be represented by 2×2 complex matrices or by 4×4 real matrices. We do not use these representations here. However, their existence serves as proof that quaternions satisfy the associative and distributive laws of algebra. The result is that the product of two quaternions is

$$\begin{aligned}(Q_0 + \boldsymbol{Q})(P_0 + \boldsymbol{P}) \\ = Q_0P_0 - \boldsymbol{Q}\cdot\boldsymbol{P} + Q_0\boldsymbol{P} + P_0\boldsymbol{Q} + \boldsymbol{P} \times \boldsymbol{Q}\end{aligned} \tag{A8}$$

The number part of the product is made of the numerical product of the two number parts and (minus) the dot product of the two vector parts; the vector part of the product is made of usual products of the vector part of each quaternion by the number part of the other and the vector product of the two vector parts.

We define the conjugate of a quaternion by

$$(Q_0 + \boldsymbol{Q})^* \equiv Q_0 - \boldsymbol{Q} \tag{A9}$$

This is quite similar to the definition of complex conjugate for complex numbers. Here, too, we find that the product of a quaternion with its conjugate is a positive definite pure number

$$QQ^* = Q_0^2 + \boldsymbol{Q}^2 \tag{A10}$$

The absolute value of a quaternion is defined as

$$|Q| \equiv (QQ^*)^{1/2} \tag{A11}$$

Note that the absolute value of a quaternion that is a pure number or a pure vector conforms to the definition of absolute value of numbers and of vectors. We call a quaternion whose absolute value is unity a "unit quaternion."

The following rules apply to absolute values and to conjugates of products of quaternions:

$$|PQR\ldots| = |P||Q||R|\ldots \tag{A12}$$

$$(PQR\ldots)^* = \ldots R^*Q^*P^* \tag{A13}$$

The absolute value of a product is the product of the absolute values. The conjugate of a product is the product of the conjugates in the reverse order.

Rotations are represented by unit quaternions which may be expressed as

$$Q = \cos(\tfrac{1}{2}\alpha) + \boldsymbol{e}\sin(\tfrac{1}{2}\alpha) \tag{A14}$$

where $\boldsymbol{e}$ is a unit vector representing the axis of the rotation, and α is the angle of the rotation in the positive sense about $\boldsymbol{e}$. The rotation is applied to a vector by the construction

$$x' = QxQ^* \tag{A15}$$

Note that the same transformation when applied to a number leaves it unchanged:

$$\lambda = Q\lambda Q^* \tag{A16}$$

In view of this, the number part of a quaternion is a scalar. The terminology "scalar part" and "vector part" may be used in place of "number part" and "vector part."

Rotations may be combined by quaternion multiplication. An infinitesimal rotation may be represented by the quaternion

$$\mathrm{d}Q = 1 + \tfrac{1}{2}\boldsymbol{\omega}\,\mathrm{d}t \tag{A17}$$

where $\boldsymbol{\omega}$ is the vector of angular velocity. Multiplying Eq. (A17) by Eq. (A14) and separating the first order infinitesimal part, we find

$$\mathrm{d}Q_0/\mathrm{d}t = -\tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{Q} \tag{A18}$$

$$\mathrm{d}\boldsymbol{Q}/\mathrm{d}t = \tfrac{1}{2}\boldsymbol{\omega}Q_0 + \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{Q} \tag{A19}$$

The last equations define the orientation history of a body in terms of quaternions when the time history of angular velocity is known.

Once the quaternion Q is known, the rotation [Eq (A15)] may be expressed without computing a single trigonometric function. In terms of quaternion components, Eq. (A26) becomes

$$x' = \{Q_0^2 - \boldsymbol{Q}^2\}x + 2(\boldsymbol{Q}\cdot x)\boldsymbol{Q} + 2Q_0\boldsymbol{Q} \times x \tag{A20}$$

The rotation matrix may be read from the last equation as

$$R_{ij} = (Q_0^2 - Q_k Q_k)\delta_{ij} + 2Q_i Q_j - 2\varepsilon_{ijk} Q_0 Q_k \quad \text{(A21)}$$

or more explicitly as

$$\boldsymbol{R} = \left\|\begin{matrix} Q_0^2 + Q_x^2 - Q_y^2 - Q_z^2 & 2Q_x Q_y - 2Q_z Q_0 & 2Q_x Q_z + 2Q_y Q_0 \\ 2Q_y Q_x + 2Q_z Q_0 & Q_0^2 - Q_x^2 + Q_y^2 - Q_z^2 & 2Q_y Q_z - 2Q_x Q_0 \\ 2Q_z Q_x - 2Q_y Q_0 & 2Q_z Q_y + 2Q_x Q_0 & Q_0^2 - Q_x^2 - Q_y^2 + Q_z^2 \end{matrix}\right\| \quad \text{(A22)}$$

Note that the quaternion representing the null rotation, obtained by substituting $\alpha = 0$ in Eq. (A15), is the number 1. But a rotation by 2π gives rise to $Q = -1$. However, the rotation induced by Q and by $-Q$ are the same. The sign cancels in Eqs. (A20), (A21), and (A22), which are all quadratic in the components of Q.

In summary, unit quaternions describe rotations in three dimensions. The quaternion corresponding to a given rotation is unique up to a sign. Once the quaternion is known, computation of the effect of the rotation requires no trigonometry. Rotations may be combined by quaternion multiplication. Quaternion history may be derived from the history of angular velocity using the differential Eqs. (A18) and (A19).

Quaternions describe the three degrees-of-freedom of rotation in three dimensions by four parameters (the quaternion components) subject to one constraint:

$$|Q|^2 = (QQ^*) = Q_0^2 + \boldsymbol{Q}\cdot\boldsymbol{Q} = 1 \quad \text{(A23)}$$

The constraint is self-preserving. The differential Eqs. (A18) and (A19) assure that the constraint, if satisfied initially, will be satisfied for all time.

When computers are used to integrate the differential equations, truncation errors can cause violation of the constraints. If unchecked, such violations would statistically grow as time increases. For this reason it is necessary to reinforce the constraints at intervals, normally at every integration step.

The requirement that a quaternion obtained from integrating Eqs. (A18) and (A19) be a unit quaternion may be reinforced by renormalization. That is, each of the four components is multiplied by

$$|Q|^{-1} = \{Q_0^2 + \boldsymbol{Q}^2\}^{-1/2} \quad \text{(A24)}$$

However, $|Q|$ is virtually equal to 1, any difference being merely the effect of truncation error for one integration step. For this reason the full computational burden of Eq. (A24) (including a square root and a division) is not necessary. Instead, one may expand in the small difference and retain terms only up to first order:

$$|Q|^{-1} = \{1 - (1 - Q_0^2 - \boldsymbol{Q}^2)\}^{-1/2}$$
$$\approx 1 + \tfrac{1}{2}(1 - Q_0^2 - \boldsymbol{Q}^2) = \tfrac{3}{2} - \tfrac{1}{2}(Q_0^2 + \boldsymbol{Q}^2) \quad \text{(A25)}$$

The last factor, which is vitually unity, suffices to enforce the constraint for all times. The burden of computing it involves only multiplications and additions.

Appendix B: Special Rotation Vectors

Special rotation vectors are three-dimensional vectors of length unity or less:

$$|\boldsymbol{R}| \leq 1 \quad \text{(B1)}$$

Each SRV defines a unique rotation as follows: The axis of the rotation is in the direction of $\boldsymbol{R}$; the angle of the rotation α is determined by

$$0 \leq \alpha \leq \pi \quad \text{(B2)}$$

$$\sin(\tfrac{1}{2}\alpha) = |\boldsymbol{R}| \quad \text{(B3)}$$

The rotation is taken in the positive screw sense around $\boldsymbol{R}$. Only the range in Eq. (B2) need be included, since rotations described by an SRV in the opposite direction cover the complementary range. The origin corresponds to no rotation (reference orientation). The points on the surface of the sphere describe rotations by π. Opposite points on the surface of the sphere, corresponding to rotations by 180 deg in opposite directions, describe the same orientation. Each pair of opposite surface points defined a single SRV. However, in practice, any one can be used.

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Total Least Squares Estimation of Aerodynamic Model Parameters from Flight Data

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Introduction

EQUATION-ERROR techniques for the identification of aircraft models from flight test data were introduced in the 1950s. Greenberg[1] applies these techniques on the problem of identifying parameters in the aircraft's differential equations-of-motion, later Gerlach[2] shows applications to linear aerodynamic equations. Both applications are based on the method of "least squares" introduced by Gauss[3] in 1809.

One way of assessing the accuracy of the estimated parameters is through consulting the least squares generated variance matrix. Another way is through repeating the experiment and observing the scatter in the estimated parameters. Both assessments are often found to disagree. The calculated variance matrix may predict a high accuracy, while the scatter in the estimated parameters suggests the opposite.

This Note shows an example where this controversy was caused by the incapability of the least squares method to

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